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A stochastic production planning problem with nonlinear cost

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Abstract

Most production planning models are deterministic and often assume a linear relation between production volume and production cost. In this paper, we investigate a production planning problem in a steel production process considering the energy consumption cost which is a nonlinear function of the production quantity. Due to the uncertain environment, the production demands are stochastic. Taking a scenario-based approach to express the stochastic demands according to the knowledge of planners on the demand distributions, we formulate the stochastic production planning problem as a mixed integer nonlinear programming (MINLP) model.

Approximated with the piecewise linear functions, the MINLP model is transformed into a mixed integer linear programming model. The approximation error can be improved by adjusting the linearization ranges repeatedly. Based on the piecewise linearization, a stepwise Lagrangian relaxation (SLR) heuristic for the problem is proposed where variable splitting is introduced during Lagrangian relaxation (LR). After decomposition, one subproblem is solved by linear programming and the other is solved by an effective polynomial time algorithm. The SLR heuristic is tested on a large set of problem instances and the results show that the algorithm generates solutions very close to optimums in an acceptable time. The impact of demand uncertainty on the solution is studied by a computational discussion on scenario generation.

Keywords

- Stochastic production planning;
- Inventory;
- Scenario-based approach;
- Stepwise Lagrangian relaxation;
- Variable splitting;
- MINLP

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1. Introduction

Iron and steel industry is an essential sector in economy. This sector consumes extensive energy since most iron and steel production operations are performed at high temperature. Energy cost accounts for a large proportion of the total production cost. Therefore, energy saving in production is of great significance for steel companies to reduce cost and stay competitive. This motivates us to study the production planning problem in a hot rolling mill with the objective to minimize the energy consumption.

In the hot rolling production process studied in this paper, steel slabs are first heated up to the required temperature in the heat furnace. Then the heated slabs are rolled, on the hot rolling mills, into hot strips according to the specification of the demands. Finally, the hot strips are temporarily stored in the strip yard waiting to be delivered to customers or to be further processed in the downstream processing stages. This integrated production process is illustrated in Fig. 1. For convenience, we will call the slabs or strips products hereafter. In practice there are capacity restrictions for heating, hot rolling as well as stock holding. The total cost of the integrated production process includes production cost, especially the cost on energy consumption, and inventory cost for finished products. For each type of product produced in a period, the cost on rolling includes a setup cost associated with the product type and a variable cost proportional to the production quantity. Due to the special feature of the heat furnace, its energy consumption cost is nonlinear with respect to the production quantity. So the production planning in this integrated steel production process needs to consider nonlinear cost.

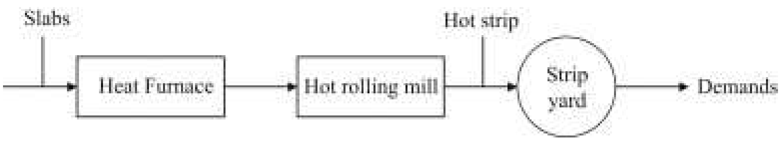

Fig. 1. The production process under consideration.

Figure options

- View in workspaceDownload full-size imageDownload as PowerPoint slideThe production process is affected by the demands of products which are full of uncertainty. The uncertainty may be caused by production plan changes in downstream stages or the random arrival of customer orders. Ultimately and to a large extent it comes from fluctuations in the world steel market. For example, demands of steel products are affected by the prices which are in turn affected by the prices of energy and raw materials such as iron ore. On the other hand, demands also have an impact on prices. Moreover, demands of steel products are also influenced by the economic situation, government policies, protectionism, etc. Because of the variety of direct and indirect impacting factors, demands can hardly be forecasted accurately. Therefore this problem needs to consider stochastic demands caused by these factors.

There has been extensive research in the general field of production planning with a large proportion concerning linear production cost. In terms of deterministic circumstances, Bahl et al. [2] provide a survey about lot sizing problems in production planning and Karimi et al. [13] provide a review of models and algorithms for single-level multi-product capacitated lot sizing problems. In addition, Xie and Dong [24] propose heuristic genetic algorithms for general capacitated lot sizing problems. Jans and Degraeve [11] review various meta-heuristics that are specially developed for lot sizing problems. Minner [14] analyzes three simple heuristics for multi-product dynamic lot sizing problem with limited warehouse capacity. In terms of stochastic circumstances, a review of models for production planning under uncertainty is given by Mula et al. [15]. Sox [19] develops an optimal solution algorithm for the single-item dynamic lot sizing problem with random demand and non-stationary costs. Bakir and Byrne [3] develop a two-stage stochastic linear programming model for the multi-product multi-period problem with stochastic demands. Haugen et al. [10] address a stochastic version of the classical W-W model (Wagner and Whitin [23]) with an extension of the backlogging possibility and develop a meta-heuristic based on progressive hedging. The model they study is single-item and no capacity constraints are considered. In recent years, Brandimarte [5] considers a stochastic version of the classical multi-item capacitated lot sizing problem and propose a heuristic algorithm based on a fix-and-relax strategy. Azaron et al. [1] develop a polynomial algorithm for the single-product multi-period lot sizing problem with concave inventory cost and linear stochastic production cost. Related to steel industry, Tang et al. [21] provide a review of planning and scheduling methods for integrated steel production. They classify the optimization methods for steel production planning and scheduling into four types and review literature associated with each type.

Production planning problems concerning nonlinear production cost are not so widely investigated as that of linear production cost. Gutiérrez et al. [9] address a single-item lot sizing problem with uncertainty demands and concave production and holding costs. Rizk et al. [17] investigate the multi-item lot sizing problems with piecewise linear resource costs. Chazal et al. [7] study the deterministic single-product production planning problem of a profit-maximizing firm with convex production and storage cost. Since the production planning problem we are facing includes multiple items with stochastic demands and nonlinear production cost, previous approaches are not applicable.

In the field of electricity generation, the generation cost is convex increasing [20], which has a similar feature to the cost in our problem. However, there are many differences between the electricity generation problem and the production planning problem in this paper. A main difference is that electricity is non-storable so that no inventory cost needs to be considered. Therefore, approaches developed for the electricity generation problem cannot be applied to our problem.

In this paper, the studied multi-item multi-period production planning problem with nonlinear production cost and stochastic demands is formulated as a MINLP model according to a scenario-based approach. The objective is to minimize both the inventory cost and the production cost. As the problem is large-scale and it is impractical to solve it by using a commercial solver, a heuristic algorithm is devised for this problem. The main difficulty in developing the heuristic is in dealing with the nonlinear cost in the objective function. Oh and Karimi [16] take a multi-segment separable programming approach to help solving the nonlinear difficulty met in their lot-sizing problem. Taking a similar approach, we propose a SLR heuristic for our problem. The proposed heuristic is tested and its effectiveness is verified through computational experiments.

The remainder of the paper is organized as follows. In the next section, the MINLP model is formulated for the problem after the energy consumption feature of the heat furnace is discussed in detail. Section 3 is devoted to devising our heuristic for the problem. An approximate formulation is developed and a SLR heuristic is presented. In Section 4, computational experiments are reported. Finally, the paper is ended with our conclusions and directions for future research in Section 5.

2. Problem description and formulation

We consider the production planning problem for the integrated production process from the heat furnace, through hot rolling, to the storage yard of rolled products. The objective is to minimize the total production and storage cost. The production cost in hot rolling and the storage cost of rolled products are typical linear functions of production quantity and inventory level, respectively, while the cost of heat furnace is mainly on energy consumption. The production plan needs to consider stochastic demands for products and backlogging is not permitted. Based on the practical production process, there is no delay between the heating process and hot rolling because heated products are charged directly into the hot rolling mill as soon as they come out of the heat furnace. The production planning problem under consideration takes a day as its planning period.

The rest of this section is organized as follows. The special feature of the energy consumption cost of the heat furnace is discussed in Section 2.1. The way of modeling stochastic demands is discussed in Section 2.2. A MINLP model for the stochastic production planning problem is formulated in Section 2.3.

2.1. Energy consumption feature of the heat furnace

There are different types of heat furnaces in the steel-rolling mill. Each type of furnace has a different relation between the energy consumption and the products quantity heated. Here, the furnace considered is of the type for which the relative distribution of heat along the furnace length is fixed. It does not change with the products quantity or the heat provided in the furnace. Meanwhile, the state of the heat furnace usually remains "on" since starting up the furnace needs a considerable cost. When there is no product in the furnace, a fixed energy consumption cost is incurred for heat preservation of the furnace. According to the work feature of this type of heat furnace, Yu et al. [25] develop the relationship between the quantity of the products heated and the energy consumption cost per period as

(1)

$E=E_0 exp(rx)$,

Here E denotes the energy consumption cost and x denotes the amount of products heated in the heat furnace per period. E_0 is a constant indicating the energy consumption cost when there is no product in the heat furnace (x=0) and r is a positive constant associated with the furnace. Equivalently, the energy consumption cost per unit of production quantity per period is

(2)

$E/x=E_0 exp(rx)/x$.

It can be seen that function E/x is convex and achieves its minimum value at $x_0=1/r$. Clearly if only the energy consumption cost in the furnace is considered, $x_0=1/r$ is the optimal production quantity.

The above analysis indicates that there is an exponential relationship between the heating quantity and the unit energy consumption in the heat furnace. Therefore it is not realistic to assume the production cost as a linear function of the production quantity in the system. To model the production planning problem accurately, we adopt the nonlinear unit energy consumption cost function (2) to express the associated production cost.

2.2. Modeling the stochastic demands

Statistical method for dealing with uncertainty is to express an uncertain parameter as a random variable with its distribution function. However, obtaining the distribution function exactly is not easy usually because of lacking enough information. Even if the exact distribution is given, the resulting stochastic optimization problem is notoriously hard to solve. Here, we use a scenario-based approach to express the uncertain demands as a set of demand scenarios according to the knowledge of the planners on the demand distributions. This is a popular approach for modeling uncertainty and has been effectively used in the stochastic unit commitment problem [20], the stochastic technology choosing and capacity planning problem [8] and many other stochastic problems.

According to this approach, the possible evolution cases of the stochastic demands throughout the planning horizon are assumed to be of finite number and represented with a set of demand scenarios. Meanwhile, each scenario is assigned a weight to reflect the probability of its realization. That is, we describe the stochastic demands as follows:

$$P\{\tilde{d}_{it} = d_{it}^{s},\ i \in I,\ t \in T\} = P_{s},\ s \in S,$$

where *S* is the set of demand scenarios, *I* is the set of products, *T* is the set of periods in the planning horizon, *dit* is the stochastic demand for product *i* in period *t*, *i*∈*I*, *t*∈*T*, and $d_{it}^{s}$is the demand for product *i* in period *t* in scenario *s*, *i*∈*I*, *t*∈*T*, *s*∈*S*, and *Ps* is the probability of scenario *s*, *s*∈*S*. To describe the evolution feature of the stochastic demands, the scenarios are modeled as a tree structure and a scenario is represented as a path from the root node to a terminal node as illustrated in Fig. 2. For a given period *t*, the demand realization in the subsequent periods is invisible and non-anticipatable. The branches in the subsequent periods may express the possible realization.

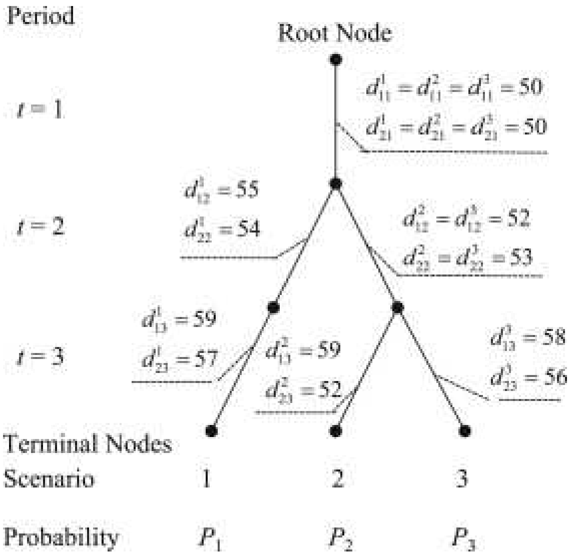

Fig. 2. An example of scenario tree for stochastic demands of two products over three periods.

Figure options

- View in workspaceDownload full-size imageDownload as PowerPoint slideFig. 2 shows that some scenarios may share the same demand path from the first period up to a certain time period. Scenarios 1, 2, and 3 share the same demand path in period 1 and $d_{11}^{1} = d_{11}^{2} = d_{11}^{3} = 50$, $d_{21}^{1} = d_{21}^{2} = d_{21}^{3} = 50$. Scenarios 2 and 3 share the same demand path from period 1 to period 2 and $d_{12}^{2} = d_{12}^{3} = 52$, $d_{22}^{2} = d_{22}^{3} = 53$. Because of the invisibility and non-anticipativity feature in the subsequent periods, scenarios sharing the same demand path before and in period *t* are indistinguishable in period *t*. In Fig. 2, scenarios 1, 2, and 3 are indistinguishable in period 1 and scenarios 2 and 3 are indistinguishable in period 2. For the indistinguishability of scenario information, we use the term, scenario bundle, mentioned by Rockafellar and Wets [18] to formulate it. A scenario bundle in a period is a set of indistinguishable scenarios in this period. Two scenarios, *s* and *j*, are members of the same bundle in period *t* if and only if $d_{i\tau}^{j} = d_{i\tau}^{s}$holds for 1≤ τ ≤*t* and all *i*. Obviously, each scenario belongs to only one bundle in a period. Let *Q*(*t*, *s*) denote the set of scenarios corresponding to the scenario bundle including scenario *s* in period *t*. Then in Fig. 2, *Q*(1, 1)=*Q*(1, 2)=*Q*(1, 3)={1, 2, 3}, *Q*(2, 1)={1}, *Q*(2, 2)=*Q*(2, 3)={2, 3}, *Q*(3, 1)={1}, *Q*(3, 2)={2}, *Q*(3, 3)={3}.

If two scenarios are indistinguishable in period *t*, the associated decisions made for these scenarios in period *t* are the same. In Fig. 2, the decisions made for scenario 2 in period 2 are the same as those for scenario 3. Based on the scenario bundle defined above, this decision feature will be reflected by introducing the indistinguishability constraints to the formulation of the problem.

2.3. The model

We first define the parameters and decision variables in the following:

Parameters

cit

unit hot rolling cost for product *i* in period *t*, $i \in I$, $t \in T$;

hit

unit inventory holding cost for product *i* in period *t* after hot rolling, $i \in I$, $t \in T$;

Seit

setup cost for product *i* in period *t*, $i \in I$, $t \in T$;

Invmax

capacity for inventory holding of the store yard after hot rolling;

$Invi_0$

initial inventory of product *i* over the planning horizon, $i \in I$;

HF_t^{max}

maximum amount of products allowed to be heated in the heat furnace in period *t*, $t \in T$;

HR_t^{max}

maximum amount of products allowed to be rolled on the hot rolling mills in period *t*, $t \in T$;

α

yield ratio from slabs to strips, i.e., the amount of strips produced by one unit amount of slabs; Its value is positive and smaller than 1;

M

a positive real number large enough.

Decision variables

x_{it}^{s}

amount of product *i* entering the heat furnace and hot rolling mills in period *t* in scenario *s*, *i*∈*I*, *t*∈*T*, *s*∈*S*;

y_{it}^{s}

amount of product *i* produced from hot rolling mills in period *t* in scenario *s*, *i*∈*I*, *t*∈*T*, *s*∈*S*;

z_{it}^{s}

a binary variable indicating whether product *i* is rolled on the hot rolling mill in period *t* in scenario *s*, *i*∈*I*, *t*∈*T*, *s*∈*S*;

Inv_{it}^{s}

inventory of product *i* at the end of period *t* in scenario *s*, *i*∈*I*, *t*∈*T*, *s*∈*S*.

In order to distinguish between the problem and its approximate version appeared in the later section, we call the problem after modeling the stochastic demands using the scenario-based approach the original problem which can be formulated as follows.

Original problem (OP)

Minimize *gO*, with

(3)

$$g_O = \sum_{s \in S} P_s \left[E_0 \sum_{t \in T} \exp\left(r \sum_{i \in I} x_{it}^{s} \right) + \sum_{i \in I} \sum_{t \in T} (c_{it} x_{it}^{s} + h_{it} Inv_{it}^{s} + Se_{it} z_{it}^{s}) \right]$$

subject to

(4)

$$Inv_{i,t-1}^{s} + y_{it}^{s} = d_{it}^{s} + Inv_{it}^{s}, \ i \in I, \ t \in T, \ s \in S$$

(5)

$$\alpha x_{it}^{s} = y_{it}^{s}, \ i \in I, \ t \in T, \ s \in S$$

(6)

$$\sum_{i \in I} Inv_{it}^{s} \le Inv^{\max}, \ t \in T, \ s \in S$$

(7)

$$\sum_{i \in I} x_{it}^{s} \le HF_t^{\max}, \ t \in T, \ s \in S$$

(8)

$$\sum_{i \in I} x_{it}^{s} \le HR_t^{\max}, \ t \in T, \ s \in S$$

(9)

$$x_{it}^{s} \le M z_{it}^{s},\ i \in I,\ t \in T,\ s \in S$$

(10)

$$x_{it}^{j} = x_{it}^{s},\ j \in Q(t, s),\ i \in I,\ t \in T,\ s \in S$$

(11)

$$y_{it}^{j} = y_{it}^{s},\ j \in Q(t, s),\ i \in I,\ t \in T,\ s \in S$$

(12)

$$z_{it}^{j} = z_{it}^{s},\ j \in Q(t, s),\ i \in I,\ t \in T,\ s \in S$$

(13)

$$Inv_{it}^{j} = Inv_{it}^{s},\ j \in Q(t, s),\ i \in I,\ t \in T,\ s \in S$$

(14)

$$x_{it}^{s} \ge 0,\ y_{it}^{s} \ge 0,\ Inv_{it}^{s} \ge 0,\ i \in I,\ t \in T,\ s \in S$$

(15)

$$z_{it}^{s} : 0\text{–}1,\ i \in I,\ t \in T,\ s \in S$$

Expression (3) offers the optimization objective. The objective is to minimize the weighted average costs under all scenarios over the planning horizon for heating, hot rolling, and inventory holding. Each weight of the cost is the probability of the associated scenario. Eqs. (4) represent the inventory balance constraints. Eqs. (5) reflect the fact that there is an output-to-input ratio for hot rolling due to the material losses in the process. Inequalities (6) represent capacity constraints of inventory holding. Inequalities (7) and (8) represent production capacity constraints for heating and hot rolling, respectively. Inequalities (9) show the variables consistency between $x_{it}^{s}$and z_{it}^{s}. Eqs. (10), (11), (12) and (13) indicate the indistinguishability constraints on decision variables. Inequalities (14) define the nonnegative value fields for the continuous variables, while constrains (15) show the binary evaluation of the integer variables.

Since *gO* increases as the production quantity increases and no backlogging is permitted, the final inventories in the optimal solution must be zero, i.e.,

(16)

$$Inv_{i|T|}^{s} = 0,\ i \in I,\ s \in S$$

where |· | is the norm of a set. Based on constraints (4), (5) and (16), we have

(17)

$$Inv_{it}^{s} = Inv_{i0} + \sum_{k \in T, k \le t} (\alpha x_{ik}^{s} - d_{ik}^{s}).$$

Replacing $y_{it}^{s}$and Inv_{it}^{s}in (3) with (5) and (17), respectively, we get

(18)

$$g_O = \sum_{s \in S} P_s \left\{ E_0 \sum_{t \in T} \exp\left(r \sum_{i \in I} x_{it}^s \right) + \sum_{i \in I} \sum_{t \in T} \left[\left(c_{it} + \alpha \sum_{k \in T, k \geq t} h_{ik} \right) x_{it}^s + Se_{it} z_{it}^s \right] \right\}$$

Here a constant term, $F = \sum_{i \in I} \sum_{t \in T} Inv_{i0} h_{it} - \sum_{s \in S} P_s [\sum_{i \in I} \sum_{t \in T} (\sum_{k \in T, k \geq t} h_{ik}) d_{it}^s]$, is omitted from (18) since it has no effect on the solution.

Compared with the traditional production planning model, the classical W-W model for example, the above model shows the following characteristics.

(1)

The W-W model considers the single-item problem and the main task is to minimize only the sum of two types of costs, order cost and stock holding cost. Our model includes multiple items, which are interconnected through the capacity constraints. Besides, our model considers the balance of more cost elements including setup cost, production cost proportional to production quantity, energy consumption cost, and inventory holding cost.

(2)

Since the energy consumption cost includes an exponential function, the objective function is nonlinear, which results in a MINLP problem and endows the problem solving with challenge.

(3)

Demands are stochastic, which is closer to the practical production. Meanwhile, the approach expressing uncertainty introduces a set of scenarios. This adds another dimension, scenario, to the model and makes it different from deterministic models. However, this approach inevitably results in a large-scale MINLP problem since the problem size increases as the number of scenarios associated with the uncertainty increases. For example, in a problem with $|T|$ periods and a scenario tree structure illustrated in Fig. 3, the number of scenarios is $2^{|T|-1}$, which increases exponentially as the planning horizon extends.

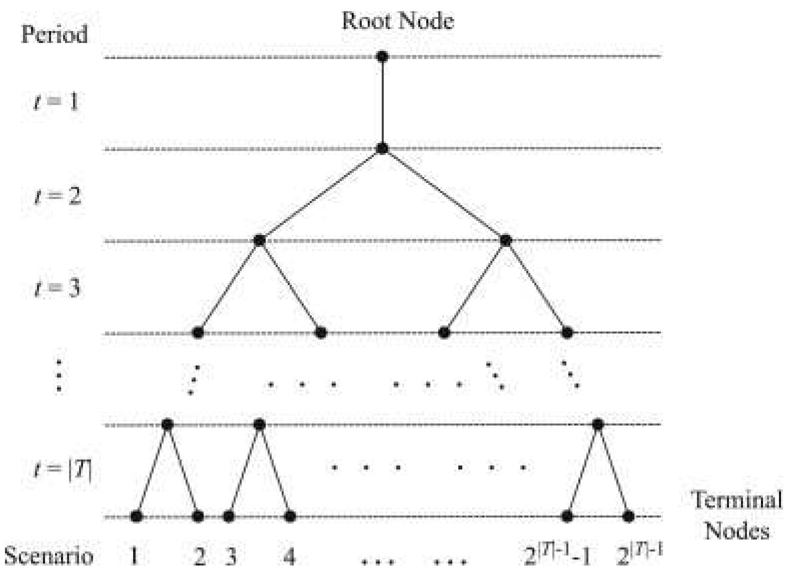

Fig. 3. An example of scenario tree structure with a $|T|$-period horizon.

Figure options

- View in workspaceDownload full-size imageDownload as PowerPoint slide3. Heuristic algorithm

In this part, a heuristic is devised for problem OP. It is a stepwise procedure alternately carrying out the linear approximation and applying the LR solution algorithm. In each cycle, the procedure includes two stages. In the first stage, the approximation range is updated according to the results of the last cycle so that a better solution may be found and the exponential term in problem OP is approximated linearly over this range. In the second stage, the approximate problem is solved using a variable splitting-based LR algorithm. Based on our heuristic, an upper bound for the optimal objective values of both the original problem and the approximate problem is presented. At the end of this part, a comparison between this paper and that of Carøe and Schultz [6] is given.

3.1. Linear approximation

To handle the nonlinear intractability of the problem, the linear approximation is adopted to over-approximate the nonlinear term $E_0\exp(r\sum_{i\in I}x_{it}^{s})$ with a piecewise linear function. First, we impose a lower bound $x_{it}^{sL}\geq 0$ and an upper bound x_{it}^{sU} on x_{it}^{s} since x_{it}^{s} is non-negative and finite. The lower and upper bounds can be found by analyzing the constraints on x_{it}^{s}. A pair of obvious bounds is $x_{it}^{sL}=0$ and $x_{it}^{sU}=H_t^{\max}$, where $H_t^{\max}=\min\{HF_t^{\max}, HR_t^{\max}\}$. Then, the interval $[\sum_{i\in I}x_{it}^{sL}, \sum_{i\in I}x_{it}^{sU}]$ is divided into $|H|$ equal subintervals with the length of $\Delta_t^s=\sum_{i\in I}\Delta_{it}^s$, where H is the set of subintervals indexed with h and $\Delta_{it}^s=(x_{it}^{sU}-x_{it}^{sL})/|H|$. Finally, over-approximating the curve $E_0\exp(r\sum_{i\in I}x_{it}^{s})$ with a piecewise linear function, we get

(19)

$$E_0\exp\left(r\sum_{i\in I}x_{it}^{s}\right)\approx E_0\exp\left(r\sum_{i\in I}x_{it}^{sL}\right)+\sum_{i\in I}\sum_{h\in H}slope_t^{sh}x_{it}^{sh}$$

where

(20)

$$x_{it}^{sh}\geq 0,\quad i\in I,\quad t\in T,\quad s\in S,\quad h\in H$$

(21)

$$x_{it}^{s}=x_{it}^{sL}+\sum_{h\in H}x_{it}^{sh},\quad i\in I,\quad t\in T,\quad s\in S,$$

$$slope_t^{sh}=E_0\exp\left(r\sum_{i\in I}x_{it}^{sL}\right)[\exp(rh\Delta_t^s)-\exp(r(h-1)\Delta_t^s)]/\Delta_t^s$$

is the slope of the piecewise linear function in the *h*th subinterval. Furthermore, to make the approximation mathematically legitimate, the following conditions must be satisfied:

(22)

$$x_{it}^{sh} \le \Delta_{it}^{s}, \quad i \in I, \quad t \in T, \quad s \in S, \quad h \in H$$

(23)

$$x_{it}^{s,h+1} = \cdots = x_{it}^{s|H|} = 0 \ \ if \ \ x_{it}^{sh} = 0, \quad i \in I, \ t \in T, \ s \in S, \ h \in H$$

(24)

$$x_{it}^{s1} = \cdots = x_{it}^{s,h-1} = \Delta_{t}^{s} \ \ if \ \ x_{it}^{sh} \ne 0, \quad i \in I, \ t \in T, \ s \in S, \ h \in H.$$

Replacing the exponential function in (18) with (19), we can obtain the approximate formulation of problem OP as follows.

Approximate problem (AP)

Minimize *gA*, with

(25)

$$g_A = \sum_{s \in S} P_s \left[\sum_{i \in I} \sum_{t \in T} \sum_{h \in H} \left(slope_t^{sh} + c_{it} + \alpha \sum_{k \in T, k \ge t} h_{ik} \right) x_{it}^{sh} + \sum_{i \in I} \sum_{t \in T} Se_{it} z_{it}^{s} \right]$$

subject to

(26)

$$Inv_{i0} + \alpha \sum_{k \in T, k \le t} x_{ik}^{s} \ge \sum_{k \in T, k \le t} d_{ik}^{s}, \quad i \in I, t \in T, t \le |T| - 1, s \in S$$

(27)

$$Inv_{i0} + \alpha \sum_{t \in T} x_{it}^{s} = \sum_{t \in T} d_{it}^{s}, \quad i \in I, \ s \in S$$

(28)

$$\sum_{i \in I} \left[Inv_{i0} + \alpha \sum_{k \in T, k \le t} x_{ik}^{s} - \sum_{k \in T, k \le t} d_{ik}^{s} \right] \le Inv^{\max}, \quad t \in T, \ s \in S$$

and constraints (7), (8), (9), (10), (12), (15), (20), (21), (22), (23) and (24).

In (25), a constant term

$$G = \sum_{s \in S} P_s \left[\sum_{t \in T} E_0 \exp\left(r \sum_{i \in I} x_{it}^{sL} \right) + \sum_{i \in I} \sum_{t \in T} \left(c_{it} + \alpha \sum_{k \ge t, k \in T} h_{ik} \right) x_{it}^{sL} \right],$$

is omitted since it has no effect on the solution.

Problem AP is NP-hard since one of its special cases, a multi-item lot-sizing problem with capacity constraints, has been proved to be NP-hard. Consequently, no polynomial time algorithm can be found to solve it exactly. For problem AP, we are going to devise a variable splitting-based LR algorithm according to the characteristics of the model.

3.2. Variable splitting-based LR

LR plays a primary role in dealing with large-scale separable mixed integer programming problems in the past decades. The key idea of this method is to relax the coupling constraints through introducing Lagrangian multipliers and decompose the complicated problem into some simple subproblems or many small-scale subproblems. Given the Lagrangian multipliers, the relaxed problem provides a lower bound for the optimal primal objective value in a minimization problem. Generally, by way of updating the Lagrangian multipliers effectively, the lower bound can be improved gradually.

To diminish the loss of information included in the coupling constraints and improve the performance of the algorithm, variable splitting is introduced into the LR algorithm. Variable splitting is an effective method for getting a stronger lower bound. Jörnsten and Näsberg [12] have used this approach in solving the generalized assignment problem. Barcia and Jörnsten [4] have improved this method by combining it with bound improving sequences. The main step of variable splitting is to transform the problem into an equivalent one through introducing artificial variables which are copies of some original decision variables. The resulting variable copy constraints are usually relaxed by the LR algorithm. By using this technique, problem AP can be solved as follows.

3.2.1. An equivalent problem

In problem AP, a set of artificial variables $\{az_{it}^{s},\ i\in I,\ t\in T,\ s\in S : az_{it}^{s}\geq 0\}$and the following variable copy constraints:

(29)

$$z_{it}^{s}=az_{it}^{s},\ i\in I,\ t\in T,\ s\in S$$

are added and an equivalent problem is obtained below.

Equivalent problem (EP)

Minimize *gE*, with

(30)

$$g_E=\sum_{s\in S}P_s\left[\sum_{i\in I}\sum_{t\in T}\sum_{h\in H}\left(slope_t^{sh}+c_{it}+\alpha\sum_{k\in T,k\geq t}h_{ik}\right)x_{it}^{sh}+\sum_{i\in I}\sum_{t\in T}Se_{it}az_{it}^{s}\right]$$

subject to

(31)

$$x_{it}^{s}\leq Maz_{it}^{s},\ i\in I,\ t\in T,\ s\in S$$

(32)

$$az_{it}^{s} \geq 0,\ i \in I,\ t \in T,\ s \in S$$

and constraints (7), (8), (10), (12), (15), (20), (21), (22), (23), (24), (26), (27), (28) and (29).

3.2.2. Relaxed problem

To give an explicit expression for the relaxed problem, we define $\pi(t, s)$ to be the scenario with the smallest index among the scenarios sharing the same bundle with scenario *s* in period *t*, i.e., $\pi(t, s) = \min\{j : j \in Q(t, s)\}$, *t*∈*T*, *s*∈*S*, and let

$$A_{tsj} = \begin{cases} 1, & \text{if } j = \pi(t, s), \\ 0, & \text{otherwise}, \end{cases} \quad t \in T, s \in S, j \in S.$$

Based on the above definitions, constraints (12) are expressed equivalently by

(33)

$$z_{it}^{s} = \sum_{j \in S} P_j z_{it}^{j} A_{tj\pi(t,s)} \Big/ \sum_{j \in S} P_j A_{tj\pi(t,s)}, \quad i \in I,\ t \in T,\ s \in S.$$

Relaxing coupling constraints (33) and (29) using Lagrangian multipliers $\mu_{1it}^{s} \in R$ and $\mu_{2it}^{s} \in R$, respectively, *i*∈*I*, *t*∈*T*, *s*∈*S*, we can generate the following relaxed problem.

(LRP)

Minimize *gLR*(μ), with

(34)

$$g_{LR}(\mu) = \sum_{s \in S} P_s \Bigg[\sum_{i \in I} \sum_{t \in T} \sum_{h \in H} \Big(slope_t^{sh} + c_{it} + \alpha \sum_{k \in T, k \geq t} h_{ik} \Big) x_{it}^{sh} + \sum_{i \in I} \sum_{t \in T} Se_{it} az_{it}^{s} + \sum_{s \in I} \sum_{t \in T} \mu_{1it}^{s} \Big(z_{it}^{s} - \sum_{j \in S} P_j z_{it}^{j} A_{tj\pi(t,s)} / \sum_{j \in S} P_j A_{tj\pi(t,s)} \Big) + \sum_{i \in I} \sum_{t \in T} \mu_{2it}^{s} (z_{it}^{s} - az_{it}^{s}) \Bigg]$$

subject to constraints (7), (8), (10), (15), (20), (21), (22), (23), (24), (26), (27), (28), (31) and (32).

Here, μ is the multiplier vector with elements μ_{qit}^{s}, *q*=1,2, *i*∈*I*, *t*∈*T*, *s*∈*S*. Problem LRP can be decomposed into two independent subproblems, LRP_1 and LRP_2, as follows.

(LRP**1**)

Minimize gLR_1(μ), with

(35)

$$g_{LR1}(\mu) = \sum_{s \in S} P_s \Bigg[\sum_{i \in I} \sum_{t \in T} \sum_{h \in H} \Big(slope_t^{sh} + c_{it} + \alpha \sum_{k \in T, k \geq t} h_{ik} \Big) x_{it}^{sh} + \sum_{i \in I} \sum_{t \in T} (Se_{it} - \mu_{2it}^{s}) az_{it}^{s} \Bigg]$$

subject to constraints (7), (8), (10), (20), (21), (22), (26), (27), (28), (31) and (32).

(LRP**2**)

Minimize $gLR_2(\mu)$, with

(36)

$$g_{LR2}(\mathbf{\mu})=\sum_{s\in S}\sum_{i\in I}\sum_{t\in T}P_s\left(\mu_{1it}^s+\mu_{2it}^s-\sum_{j\in S}\mu_{1it}^j P_j A_{tsm(t,j)}/\sum_{j\in S}P_j A_{tsm(t,j)}\right)z_{it}^s$$

subject to constraints (15).

Problem LRP_1 is a linear programming problem and can be solved optimally using standard optimization software. Since $\exp(r\sum_{i=1}^{n}x_{it})$ is convex, there must be $slope_t^{s1}<slope_t^{s2}<\cdots<slope_t^{sH}$. Therefore constraints (23) and (24) can be satisfied naturally by any solution to problem LRP_1 and are omitted from the problem. Since z_{it}^s is a binary variable, problem LRP_2 can be easily solved by using the following approach.

Step 1. Compute $B_{it}^s=\mu_{1it}^s+\mu_{2it}^s-\sum_{j\in S}\mu_{1it}^j P_j A_{tsm(t,j)}/\sum_{j\in S}P_j A_{tsm(t,j)}$, *i*∈*I*, *t*∈*T*, *s*∈*S*.

Step 2. Let $z_{it}^s=1$ if $B_{it}^s<0$ and $z_{it}^s=0$ otherwise, *i*∈*I*, *t*∈*T*, *s*∈*S*.

3.2.3. Construction of a feasible solution

The solution to the relaxed problem is usually infeasible to problem EP due to the relaxation of the coupling constraints. Fortunately, the relaxation solution can be easily recovered to be feasible based on the optimal solution to problem LRP_1 by letting $az_{it}^s=z_{it}^s=1$ if $x_{it}^s>0$ and $az_{it}^s=z_{it}^s=0$ otherwise. Taking the obtained feasible solution $X=\{x_{it}^s, az_{it}^s, z_{it}^s, i\in I, t\in T, s\in S\}$ as the initial solution, we adjust the decision variables according to the following strategy to improve the solution quality.

Step 1. Set $\bar{t}=T$.

Step 2. Set $\bar{s}=1$ and *Vs*=∅, where *Vs* is the set of visited scenarios in period $\bar{t}$.

Step 3. Adjust the production quantities between period $\bar{t}$ and period $\bar{t}-1$ without violating the indistinguishability constraints. If $x_{i\bar{t}}^{\bar{s}}>0$ and $x_{i,\bar{t}-1}^{\bar{s}}>0$, let

$$\bar{x}_{it}^s=\begin{cases}x_{i\bar{t}}^s-x_{i\bar{t}}^{\bar{s}}, & \text{if } t=\bar{t} \text{ and } s\in Q(\bar{t}-1,\bar{s}),\\ x_{it}^s, & \text{otherwise,}\end{cases}$$

$$\bar{x}_{i,t-1}^s=\begin{cases}x_{i,\bar{t}-1}^s+x_{i\bar{t}}^{\bar{s}}, & \text{if } t=\bar{t} \text{ and } s\in Q(\bar{t}-1,\bar{s}),\\ x_{i,t-1}^s, & \text{otherwise,}\end{cases}$$

i∈*I*, *t*∈*T*, *s*∈*S*, where *Q*(*t*, *s*) is the scenario bundle that includes scenario *s* in period *t*, as defined in Section 2.2. If variables, $\bar{x}_{it}^s, i\in I, t\in T, s\in S$, are feasible to problem EP, let

$$\overline{az}_{it}^s=\bar{z}_{it}^s=\begin{cases}1, & \text{if } \bar{x}_{it}^s>0,\\ 0, & \text{otherwise,}\end{cases}\quad i\in I, t\in T, s\in S.$$

Step 4. If $\overline{X}=\{\overline{x}_{it}^{s}, \overline{z}_{it}^{s}, a\overline{z}_{it}^{s}, i\in I, t\in T, s\in S\}$can offer a lower upper bound for problem EP, update *X* with $\overline{X}$. Update *Vs* with $Vs\cup\{\overline{s}\}$.

Step 5. If $\overline{s}=|S|$, go to *Step* 6. Otherwise, set $\overline{s}=\overline{s}+1$.

Step 6. If there exists some scenario *j*∈*Vs* satisfying $A_{\overline{t}\overline{s}j}=1$, update *Vs* with $Vs\cup\{\overline{s}\}$and go to S*tep* 4. Otherwise, go to S*tep* 3.

Step 7. If $t=2$, stop. Otherwise, set $t=t-1$and go to *Step* 2.

The obtained feasible solution can provide an upper bound for problem EP.