Problem statement

Sailco Corporation must determine how many sailboats should be produced during each of the next four quarters (one quarter = three months). The demand during each of the next four quarters is as follows: first quarter, 40 sailboats; second quarter, 60 sailboats; third quarter, 75 sailboats; fourth quarter, 25 sailboats. Sailco must meet demands on time. At the beginning of the first quarter, Sailco has an inventory of 10 sailboats. At the beginning of each quarter, Sailco must decide how many sailboats should be produced during that quarter. For simplicity, we assume that sailboats manufactured during a quarter can be used to meet demand for that quarter. During each quarter, Sailco can produce up to 40 sailboats with regular-time labour at a total cost of $400 per sailboat. By having employees work overtime during a quarter, Sailco can produce additional sailboats with overtime labor. Overtime labour is split into two categories: the total cost for the first 10 sailboats produced on overtime is $450 per sailboat; the next 10 sailboats cost $500 per sailboat. A maximum of 60 sailboats can be produced per quarter. Sailco is capable of selling each sailboat demanded during each quarter at $500 per sailboat.

At the end of each quarter (after production has occurred and the current quarter's demand has been satisfied), a carrying or holding cost of $20 per sailboat is incurred.

Use i) linear programming and ii) integer programming to determine a production schedule to minimize the sum of production and inventory costs during the next four quarters. iii) Use integer programming to maximize the profit made by Sailco at the end of the fourth quarters.

LP FORMULATION

i) Linear programming

Decision variable

$$x_t = the\ number\ of\ sailboats\ produced\ during\ period\ t\ using\ regular\ time\ labour$$
$$\text{(at \$400 per sailboat)}$$

$$y_t = the\ number\ of\ sailboats\ produced\ during\ period\ t\ using\ overtime\ labour$$
$$\text{(at \$450 per sailboat)}$$

$$z_t = the\ number\ of\ sailboats\ produced\ during\ period\ t\ using\ extra\ overtime\ labour$$
$$\text{(at \$500 per sailboat)}$$

$$t = 1, 2, 3, 4$$

Define the number of sailboats on hand at the end of quarter *t* as:

$$i_t = \text{ number of sailboats on hand at end of quarter } t$$

$$t = 1, 2, 3, 4$$

Objective function

$$\begin{aligned}\text{Min z} = \ &\text{cost of regulat time producing} + \text{cost of overtime producing}\\ &+ \text{cost of extra overtime producing} + \text{holding cost}\end{aligned}$$

$$\begin{aligned}\text{Min z} = \ &400x_1 + 400x_2 + 400x_3 + 400x_4\\ &+ 450y_1 + 450y_2 + 450y_3 + 450y_4\\ &+ 500z_1 + 500z_2 + 500z_3 + 500z_4\\ &+ 20i_1 + 20i_2 + 20i_3 + 20i_4\end{aligned}$$

Subject to constraints

For quarter t:

$$\text{Inventory at end of quarter } t = i\text{nventory at the end of quarter } \ t-1$$
$$+q\text{uarter } t \text{ production} - \text{quarter } t \text{ demand}$$

$$i_t = \ \ i_{t-1} \ + x_t + y_t + z_t - d_t$$

$$t = 1, 2, 3, 4$$

Demand constraint:

At the end of the period these must be enough sailboats in inventory to satisfy the demand for that period.

$$i_t \geq 0$$

$$t = 1, 2, 3, 4$$

Capacity constraint:

A maximum of 40 sailboats can be produced using regular time labour

$$x_t \leq 40$$

Only 10 sailboats can be produced using overtime labour, and only 10 sailboats can be produced using extra overtime labour

$$y_t \leq 10$$

$$z_t \leq 10$$

It should be noted that the combination of the regular time constraint, the overtime constraint, and the maximum production constraint renders the constraint for $z_t \leq 10$ redundant. Its presence in the LP only serves to reduce the number iterations required to arrive at an optimal solution.

At most 60 sailboats can be produced per quarter.

$$x_t + y_t + z_t \leq 60$$

$$t = 1, 2, 3, 4$$

Quarterly demand and inventory levels

Period (t)	Inventory i_t	Demand d_t
0	10	0
1	$10 + x + y + z - 40$	40
2	$i_{t-1} \; + x + y + z - 60$	60
3	$i_{t-1} \; + x + y + z - 75$	75
4	$i_{t-1} \; + x + y + z - 25$	25

The final LP can be formulated as follows:

$$\begin{aligned}\text{Min z} = \; & 400x_1 + 400x_2 + 400x_3 + 400x_4 \\ & + 450y_1 + 450y_2 + 450y_3 + 450y_4 \\ & + 500z_1 + 500z_2 + 500z_3 + 500z_4 \\ & + 20i_1 + \; 20i_2 + \; 20i_3 + \; 20i_4\end{aligned}$$

s.t. $i_t = i_{t-1} + \; x_t + y_t + \; z_t - \; d_t$ Demand constraint

$x_t \leq 40$ Regular time capacity constraint

$y_t \leq 10$ Overtime capacity constraint

$x_t + y_t + z_t \leq 60$ Capacity constraint

$x_t + y_t + z_t \geq 0$ Sign restriction

$i_t \; \geq 0$ Sign restriction

$t = 1, 2, 3, 4$

Using any method, this LP can be solved to yield an optimal solution to minimize the cost of production.

IPFORMULATION

ii) Integer programming

Decision variable:

$$x_t = the\ number\ of\ sailboats\ produced\ during\ period\ t$$

$$t = 1, 2, 3, 4$$

Define the number of sailboats on hand at the end of quarter *t* as:

$$i_t = \text{ number of sailboats on hand at end of quarter } t$$

$$t = 1, 2, 3, 4$$

Objective function:

$$\text{Min z} = production\ cost + holding\ cost$$

Production cost function

$$c\ \ x_t \ = \begin{array}{ll} 400x_t & for\ \ 0 \le x_t \le 40 \\ 450x_t - 2000 & for\ 40 \le x_t \le 50 \\ 500x_t - 4500 & for\ 50 \le x_t \le 60 \end{array}$$

Holding cost function:

$$f\ \ i_t\ \ = 20i_t$$

$$\text{Min z} = \sum_{t=1}^{4} \text{c}\ \ x_t\ \ + f(i_t)$$

Subject to constraints:

For quarter t:

Demand constraint:

The demand for each period must be met. This means that at the end of the period there must be enough sailboats in inventory to satisfy the demand for that period. The inventory is calculated as the sum of the inventory left over from the previous period and the sailboats produced during this period less the demand.

$$\text{Inventory at end of quarter } t = i\text{nventory at the end of quarter } \; t-1 \\ + q\text{uarter } t \text{ production} - \text{quarter } t \text{ demand}$$

$$i_t = \; i_{t-1} \; + x_t - d_t$$

$$t = 1, 2, 3, 4$$

The constraint ensures that there is always enough stock in inventory to meet the demand.

Capacity constraint:

At most 60 sailboats can be produced per period, using any combination of standard time, overtime and extra overtime.

$$x_t \leq 60$$

$$t = 1, 2, 3, 4$$

Quarterly demand and inventory levels

Period (t)	Inventory i_t	Demand d_t
0	10	0
1	$10 + x - 40$	40
2	$i_{t-1} \; + x - 60$	60
3	$i_{t-1} \; + x - 75$	75
4	$i_{t-1} \; + x - 25$	25

Formulate optimization problem:

$$\text{Min z} = \text{c}\ x_1 \ + \text{c}\ x_2 \ + \text{c}\ x_3 \ + \text{c}\ x_4 \ + f\ i_1 \ + f\ i_2 \ + f\ i_3 \ + f(i_4)$$

s.t. $i_t = i_{t-1} + x_t - d_t$ Demand constraint for quarter t

$x_t \leq 60$ Capacity constraint for quarter t

$x_t \geq 0$ Sign restriction

$i_t \geq 0$ Sign restriction

$t = 1, 2, 3, 4$

Because c(x) is a piecewise linear function, the objective function is not a linear function of x, and this optimization problem is not an LP. By using the method described below, however, we can transform this problem into an IP. Knowing that the break points for c(x) are 0, 40, 50, and 60, we proceed as follows:

A piecewise linear function is not a linear function, thus general techniques used to solve linear problems, can't be applied. By using 0-1 variables, we are able to transform a piecewise linear function and represent it as a linear function.

Step 1: Replace c(x_t) by $c\ x_t \ = z_{t1}c\ 0 \ + z_{t2}c\ 40 \ + z_{t3}c\ 50 \ + z_{t4}c(60)$

Wherever c(x_t) appears in the optimization problem, replace this by a new variable z_{ti} multiplied by the cost calculated at each break point. Thus, for this problem, we find the expression above.

Step 2: Add the constraints:

The constraints listed below are a combination of 0-1 variables and the variable z_{ti} added earlier in order to represent the piecewise linear function as a linear function.

$$x_t = z_{t1}\ 0 \ + z_{t2}\ 40 \ + z_{t3}\ 50 \ + z_{t4}(60)$$

This constraint was added in step 1. This transforms the piecewise linear function into a single, linear function. This, however, is not sufficient in itself therefor we add the following constraints as well:

$$z_{t1} \leq y_{t1}$$
$$z_{t2} \leq y_{t1} + y_{t2}$$
$$z_{t3} \leq y_{t2} + y_{t3}$$
$$z_{t4} \leq y_{t3}$$

The variable y_{ti} is a binary variable (0-1 variable) which serves as a "yes-no" variable. The variable z_{ti} is a linearization variable which ensures that the piecewise linear function can be transformed into a linear function. These constraints will be used during the solution of the problem to decide during which costing period (for each quarter) the sailboats will be produced.

$$z_{t1} + z_{t2} + z_{t3} + z_{t4} = 1$$
$$y_{t1} + y_{t2} + y_{t3} = 1$$

These constraints limit the sum of all the binary variables and the sum of the linearization variables to equal one. For the binary variables this means, for each quarter, there can only be one y_{ti} variable equal to one. The z_{ti} variable is limited only in sign and not by value, therefor it can be any value larger than or equal to zero (see below); however the sum of these variables for each quarter must equal 1, which limits each variable to a fraction. This leads to the last of the linearization constraints:

$$y_{ti} = 0 \text{ or } 1 \quad i = 1, 2, 3$$
$$z_{ti} \geq 0 \quad i = 1, 2, 3, 4$$
$$t = 1, 2, 3, 4$$

These constraints have been specifically adapted to be applicable to this problem, but can be changed for any piecewise linear problem.

This problem is made much more complex due to the fact that multiple periods are taken into the equation. In order to formulate the problem we will initially consider only the first period.

For the first quarter, t = 1:

$$\text{Min z} = \text{c } x_1 \ + f \ i_1$$

s.t. $i_1 = 10 + x_1 - 40$ Demand constraint for quarter 1

$x_1 \leq 60$ Capacity constraint for quarter 1

The new formulation is the following IP:

$$\text{Min z} = z_1 c\ \ 0\ \ + z_2 c\ \ 40\ \ + z_3 c\ \ 50\ \ + z_4 c(60) + f\ \ i_1$$
$$= 0z_1\ + 16000\, z_2\ + 20500\, z_3 + 25500\, z_4\ +\ 20 i_1$$

s.t. $i_1 = 10 +\ x_1 -\ 40$ Demand constraint for quarter 1

$x_1 \leq 60$ Capacity constraint for quarter 1

$x_1 =\ 0z_{11} + 40z_{12} + 50z_{13} + 60z_{14}$

$z_{11}\ \leq\ y_{11}$

$z_{12}\ \leq\ y_{11}\ +\ y_{12}$

$z_{13}\ \leq\ y_{12}\ +\ y_{13}$

$z_{14}\ \leq\ y_{13}$

$z_{11} + z_{12} + z_{13} + z_{14} = 1$

$y_{11} + y_{12} + y_{13} = 1$

$y_{1i} = 0 \text{ or } 1 \quad i = 1, 2, 3$

$z_{1i} \geq 0 \quad i = 1, 2, 3, 4$

$x_1 \geq 0$

$t = 1, 2, 3, 4$

For all four quarters the IP can be written symbolically as follows:

Objective function:

$$\text{Min z} = \sum_{t=1}^{4} \text{c } x_t \; + f(i_t)$$

Subject to constraints:

$i_t = i_{t-1} + x_t - d_t$ Demand constraint for quarter t

$x_t \le 60$ Capacity constraint for quarter t

$x_t = z_{t1}\ 0\ + z_{t2}\ 40\ + z_{t3}\ 50\ + z_{t4}(60)$

$z_{t1} \le y_{t1}$

$z_{t2} \le y_{t1} + y_{t2}$

$z_{t3} \le y_{t2} + y_{t3}$

$z_{t4} \le y_{t3}$

$z_{t1} + z_{t2} + z_{t3} + z_{t4} = 1$

$y_{t1} + y_{t2} + y_{t3} = 1$

$y_{ti} = 0 \text{ or } 1 \quad i = 1, 2, 3$

$z_{ti} \ge 0 \quad i = 1, 2, 3, 4$

$x_t \ge 0$ Sign restriction

$t = 1, 2, 3, 4$

Results:

Optimal solution: Min z = $ 79 700.00

Labor utilization per quarter:

Quarter	Production volume	Regular time	Overtime	Extra overtime
1	50	40	10	0
2	55	40	10	5
3	60	40	10	10
4	25	25	0	0

iii) Maximizing profit using IP

This problem statement greatly simplifies finding the maximum profit to be made by Sailco because we can assume the only sales made are the amounts demanded each quarter. Changing the IP from a minimisation to a maximisation problem requires only a change in the objective function as follows:

$$\text{Max z} = \sum_{t=1}^{4} 500d_t - \; c\;x_t \; + f(i_t)$$

By subtracting the cost from the sales we can determine what the maximum profit will be for this given problem statement. The complete IP will look as follows:

Objective function:

$$\text{Max z} = \sum_{t=1}^{4} 500d_t - \; c\;x_t \; + f(i_t)$$

Subject to constraints:

$i_t = i_{t-1} + x_t - d_t$ Demand constraint for quarter t

$x_t \leq 60$ Capacity constraint for quarter t

$x_t = z_{t1}\;0\; + z_{t2}\;40\; + z_{t3}\;50\; + z_{t4}(60)$

$z_{t1} \leq y_{t1}$

$z_{t2} \leq y_{t1} + y_{t2}$

$z_{t3} \leq y_{t2} + y_{t3}$

$z_{t4} \leq y_{t3}$

$z_{t1} + z_{t2} + z_{t3} + z_{t4} = 1$

$y_{t1} + y_{t2} + y_{t3} = 1$

$y_{ti} = 0 \text{ or } 1 \quad i = 1, 2, 3$

$z_{ti} \geq 0 \quad i = 1, 2, 3, 4$

$x_t \geq 0$ Sign restriction

$t = 1, 2, 3, 4$