For quarter *t*:

Demand constraint:

The demand for each period must be met. This means that at the end of the period there must be enough sailboats in inventory to satisfy the demand for that period. The inventory is calculated as the sum of the inventory left over from the previous period and the sailboats produced during this period less the demand.

$$\text{Inventory at end of quarter } t = i\text{nventory at the end of quarter } \; t-1$$
$$+q\text{uarter } t \text{ production} - \text{quarter } t \text{ demand}$$

$$i_t = \; i_{t-1} \; + x_t - d_t$$

$$t = 1, 2, 3, 4$$

The constraint ensures that there is always enough stock in inventory to meet the demand.

Capacity constraint:

At most 60 sailboats can be produced per period, using any combination of standard time, overtime and extra overtime.

$$x_t \leq 60$$

$$t = 1, 2, 3, 4$$

Quarterly demand and inventory levels

Period (t)	Inventory i_t	Demand d_t
0	10	0
1	$10 + x - 40$	40
2	$i_{t-1} \; + x - 60$	60
3	$i_{t-1} \; + x - 75$	75
4	$i_{t-1} \; + x - 25$	25

Formulate optimization problem:

$$\text{Min } z = c(x_1) + c(x_2) + c(x_3) + c(x_4) + f(i_1) + f(i_2) + f(i_3) + f(i_4)$$

s.t.

$i_t = i_{t-1} + x_t - d_t$	Demand constraint for quarter t
$x_t \leq 60$	Capacity constraint for quarter t
$x_t \geq 0$	Sign restriction
$i_t \geq 0$	Sign restriction
$t = 1, 2, 3, 4$	

Because c(x) is a piecewise linear function, the objective function is not a linear function of x, and this optimization problem is not an LP. By using the method described below, however, we can transform this problem into an IP. Knowing that the break points for c(x) are 0, 40, 50, and 60, we proceed as follows:

A piecewise linear function is not a linear function, thus general techniques used to solve linear problems, can't be applied. By using 0-1 variables, we are able to transform a piecewise linear function and represent it as a linear function.

Step 1: Replace $c(x_t)$ by $c(x_t) = z_{t1}c(0) + z_{t2}c(40) + z_{t3}c(50) + z_{t4}c(60)$

Wherever $c(x_t)$ appears in the optimization problem, replace this by a new variable z_{ti} multiplied by the cost calculated at each break point. Thus, for this problem, we find the expression above.

Step 2: Add the constraints:

The constraints listed below are a combination of 0-1 variables and the variable z_{ti} added earlier in order to represent the piecewise linear function as a linear function.

$$x_t = z_{t1}(0) + z_{t2}(40) + z_{t3}(50) + z_{t4}(60)$$

This constraint was added in step 1. This transforms the piecewise linear function into a single, linear function. This, however, is not sufficient in itself therefor we add the following constraints as well:

$$z_{t1} \leq y_{t1}$$
$$z_{t2} \leq y_{t1} + y_{t2}$$
$$z_{t3} \leq y_{t2} + y_{t3}$$
$$z_{t4} \leq y_{t3}$$

The variable y_{ti} is a binary variable (0-1 variable) which serves as a “yes-no” variable. The variable z_{ti} is a linearization variable which ensures that the piecewise linear function can be transformed into a linear function. These constraints will be used during the solution of the problem to decide during which costing period (for each quarter) the sailboats will be produced.

$$z_{t1} + z_{t2} + z_{t3} + z_{t4} = 1$$
$$y_{t1} + y_{t2} + y_{t3} = 1$$

These constraints limit the sum of all the binary variables and the sum of the linearization variables to equal one. For the binary variables this means, for each quarter, there can only be one y_{ti} variable equal to one. The z_{ti} variable is limited only in sign and not by value, therefor it can be any value larger than or equal to zero (see below); however the sum of these variables for each quarter must equal 1, which limits each variable to a fraction. This leads to the last of the linearization constraints:

$$y_{ti} = 0 \text{ or } 1 \quad i = 1, 2, 3$$
$$z_{ti} \geq 0 \quad i = 1, 2, 3, 4$$
$$t = 1, 2, 3, 4$$

These constraints have been specifically adapted to be applicable to this problem, but can be changed for any piecewise linear problem.

This problem is made much more complex due to the fact that multiple periods are taken into the equation. In order to formulate the problem we will initially consider only the first period.

For the first quarter, t = 1:

	$\text{Min z} = \text{c } x_1 + f \ i_1$	
s.t.	$i_1 = 10 + x_1 - 40$	Demand constraint for quarter 1
	$x_1 \leq 60$	Capacity constraint for quarter 1

The new formulation is the following IP:

$$\text{Min z} = z_1 c\ 0\ + z_2 c\ 40\ + z_3 c\ 50\ + z_4 c(60) + f\ i_1$$
$$= 0z_1\ + 16000\, z_2\ + 20500\, z_3 + 25500\, z_4\ +\ 20 i_1$$

s.t.

$i_1 = 10 + x_1 - 40$ Demand constraint for quarter 1

$x_1 \leq 60$ Capacity constraint for quarter 1

$x_1 = 0z_{11} + 40z_{12} + 50z_{13} + 60z_{14}$

$z_{11} \leq y_{11}$

$z_{12} \leq y_{11} + y_{12}$

$z_{13} \leq y_{12} + y_{13}$

$z_{14} \leq y_{13}$

$z_{11} + z_{12} + z_{13} + z_{14} = 1$

$y_{11} + y_{12} + y_{13} = 1$

$y_{1i} = 0 \text{ or } 1 \quad i = 1, 2, 3$

$z_{1i} \geq 0 \quad i = 1, 2, 3, 4$

$x_1 \geq 0$

$t = 1, 2, 3, 4$

For all four quarters the IP can be written symbolically as follows:

Objective function:

$$\text{Min z} = \sum_{t=1}^{4} \text{c}\ x_t\ + f(i_t)$$

Subject to constraints:

$i_t = i_{t-1} + x_t - d_t$	Demand constraint for quarter t
$x_t \leq 60$	Capacity constraint for quarter t
$x_t = z_{t1}\ 0\ + z_{t2}\ 40\ + z_{t3}\ 50\ + z_{t4}(60)$	
$z_{t1} \leq y_{t1}$	
$z_{t2} \leq y_{t1} + y_{t2}$	
$z_{t3} \leq y_{t2} + y_{t3}$	
$z_{t4} \leq y_{t3}$	
$z_{t1} + z_{t2} + z_{t3} + z_{t4} = 1$	
$y_{t1} + y_{t2} + y_{t3} = 1$	
$y_{ti} = 0 \text{ or } 1 \quad i = 1, 2, 3$	
$z_{ti} \geq 0 \quad i = 1, 2, 3, 4$	
$x_t \geq 0$	Sign restriction
$t = 1, 2, 3, 4$	