620-362
Applied Operations Research

Piecewise Linear Models

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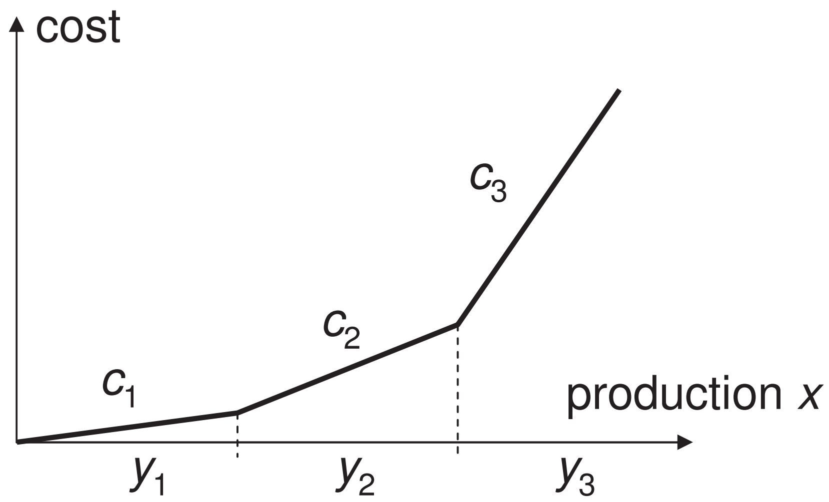
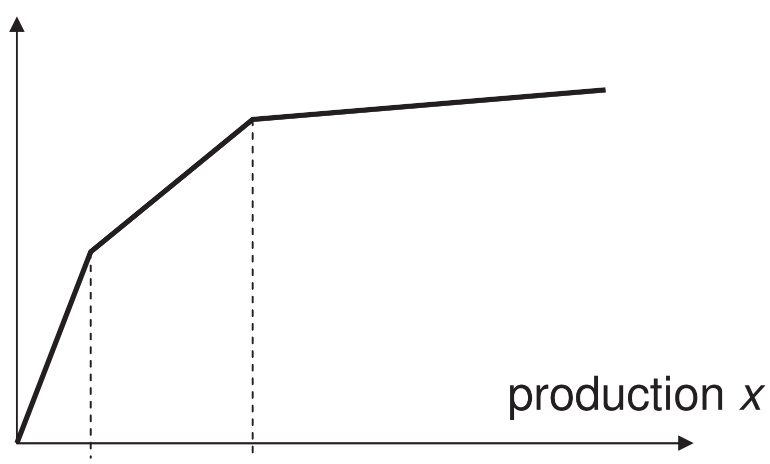
Some contents of this presentation are adapted from year 2005 course notes for 620-362 Applied Operations Research, Department of Mathematics and Statistics, The University of Melbourne (compiled by Prof Natashia Boland and Dr Renata Sotirov)

Piecewise Linear LP Models





e.g. overtime costs, tax effects

↓

minimise convex/maximise concave piecewise linear function

↓

EASY! LP-model

e.g. economies of scale

↓

minimise non-convex or concave/maximise non-concave or convex piecewise linear function

↓

HARD: Integer Programming Model

Production Scheduling

Machine	Max. # bottles able to produce per month	Labour utilization per bottle per month
1	a_1	b_1
2	a_2	b_2

Planning period: $t = 1,2,\ldots,12$
Demand for bottles: d_t, $t = 1,2,\ldots,12$
Labour availability: L_t, $t = 1,2,\ldots,12$
Initial inventory: I_0

Objective: Determine production levels on each machine in each month so as to meet demand, and minimise maximum monthly fluctuation in labour utilization.

Production Scheduling

Primary decision variables:

x_{1t} = # bottles produced on machine 1 in month t,

x_{2t} = # bottles produced on machine 2 in month t,

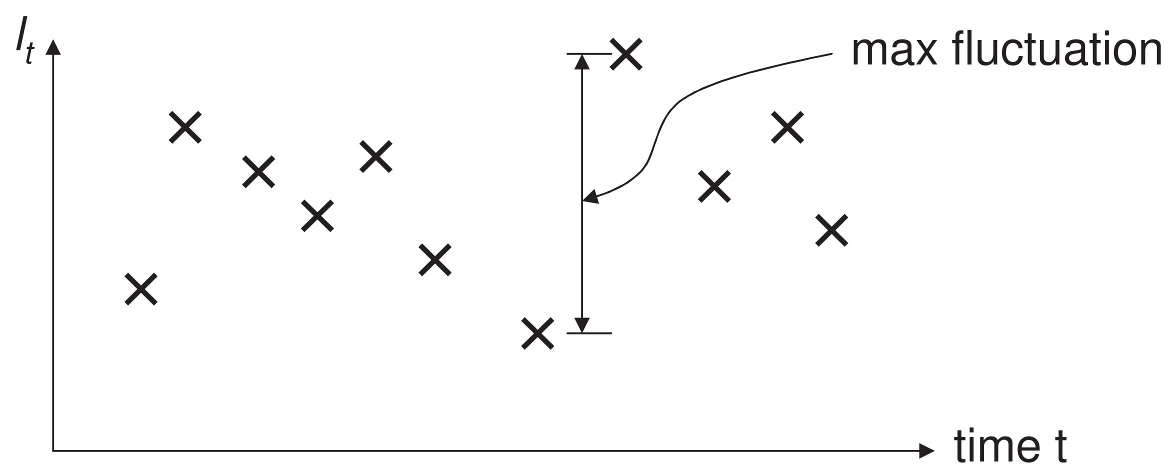
$t = 1,2,\ldots,12$

Inventory variables:

y_t = # bottles in inventory at end of month t

Labour utilization variables:

l_t = labour utilization in month t



Production Scheduling

$$\min \max_{t=2,\ldots,12} \left| l_t - l_{t-1} \right|$$

Objective function is NOT LINEAR!

$$
\begin{aligned}
s.t. \quad & l_t = b_1 x_{1t} + b_2 x_{2t}, \ \forall t = 1,\ldots,12 \\
& y_1 = x_{11} + x_{21} + I_0 - d_1 \\
& y_t = x_{1t} + x_{2t} + y_{t-1} - d_t, \quad \forall t = 2,\ldots,12 \\
& 0 \le x_{1t} \le a_1, \quad \forall t = 1,\ldots,12 \\
& 0 \le x_{2t} \le a_2, \quad \forall t = 1,\ldots,12 \\
& 0 \le l_t \le L_t, \quad \forall t = 1,\ldots,12 \\
& y_t \ge 0, \quad \forall t = 1,\ldots,12
\end{aligned}
$$

Production Scheduling

Introduce new variable: $m = \max_{t=2,\ldots,12} |l_t - l_{t-1}|$

i.e.

$$\left\{ \begin{array}{lll} \min & m & \\ s.t. & m \geq |l_t - l_{t-1}|, & \forall t = 2,\ldots,12 \\ & l_t = b_1 x_{1t} + b_2 x_{2t}, & \forall t = 1,\ldots,12 \\ & \vdots & \\ & etc & \end{array} \right.$$

Still NOT LINEAR!

Production Scheduling

Consider this: $w \geq |v| \quad \Leftrightarrow \quad w \geq v \quad \text{and} \quad w \geq -v$

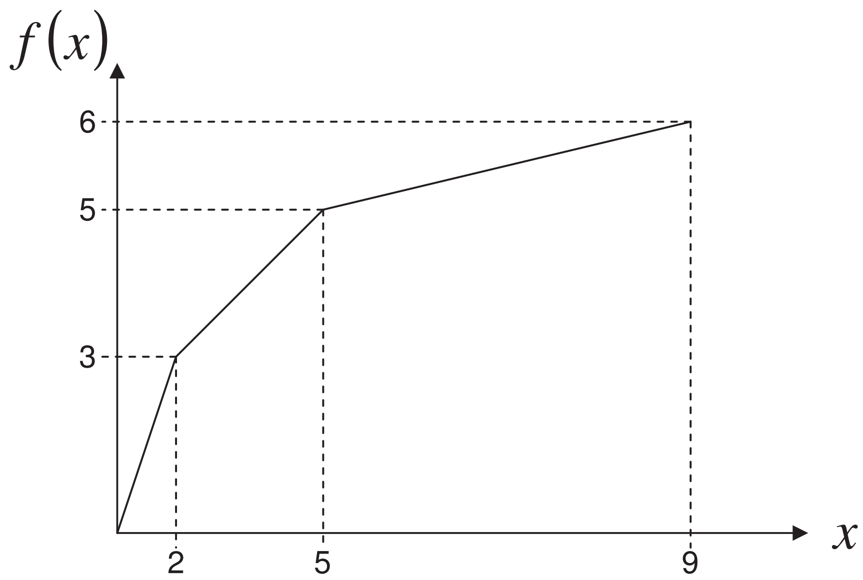
LP Model:

$$\begin{cases} \min & m \\ s.t. & \left.\begin{array}{l} m \geq l_t - l_{t-1} \\ m \geq l_{t-1} - l_t \end{array}\right\}, \quad \forall t = 2, \ldots, 12 \\ & l_t = b_1 x_{1t} + b_2 x_{2t}, \quad \forall t = 1, \ldots, 12 \\ & \vdots \\ & etc \end{cases}$$

Max. a Piecewise Linear Concave Function

Example:

$$\begin{cases} \max & f(x_1) + 2x_2 \\ s.t. & x_1 + 3x_2 \le 15 \\ & x_1 + x_2 \le 9 \\ & x_1, x_2 \ge 0 \end{cases} \qquad \text{where } f(x) = \begin{cases} \frac{3}{2}x, & 0 \le x \le 2 \\ \frac{1}{3}(2x+5), & 2 < x \le 5 \\ \frac{1}{4}(x+15) & 5 < x \le 9 \end{cases}$$



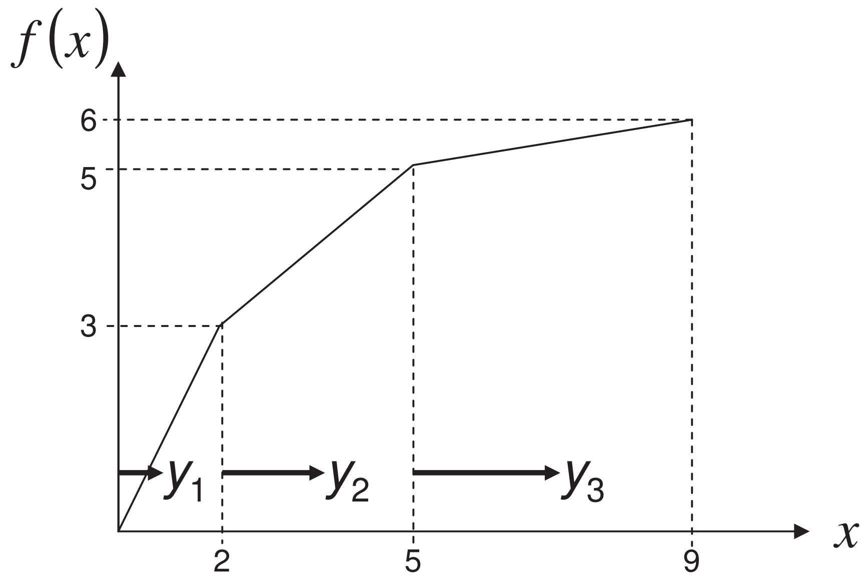
Max. a Piecewise Linear Concave Function

We can re-model this problem as an LP by replacing

$$x_1 = y_1 + y_2 + y_3$$

in all constraints, where

$$0 \le y_1 \le 2$$
$$0 \le y_2 \le 3$$
$$0 \le y_3 \le 4$$



and replacing $f(x_1)$ in the objective by

$$3/2y_1 + 2/3y_2 + 1/4y_3$$

Max. a Piecewise Linear Concave Function

New model is:

$$\begin{cases} \max & \frac{3}{2}y_1 + \frac{2}{3}y_2 + \frac{1}{4}y_3 + 2x_2 \\ s.t. & y_1 + y_2 + y_3 + 3x_2 \le 15 \\ & y_1 + y_2 + y_3 + x_2 \le 9 \\ & 0 \le y_1 \le 2 \\ & 0 \le y_2 \le 3 \\ & 0 \le y_3 \le 4 \\ & x_2 \ge 0 \end{cases}$$

Max. a Piecewise Linear Concave Function

For the new model to be correct, we need the optimal solution ($\mathbf{y}^*$, x^*_2) to satisfy:

1. if $y^*_1 < 2$ then $y^*_2 = y^*_3 = 0$, and
2. if $y^*_2 < 3$ then $y^*_3 = 0$

Can we be sure of this?

We will show by contradiction.

Max. a Piecewise Linear Concave Function

Suppose $(\boldsymbol{y}^*, x^*_2)$ is optimal,

and $y^*_1 < 2$ and $y^*_2 > 0$.

Set $\varepsilon = \min\{y^*_2, 2 - y^*_1\}$. Note $\varepsilon > 0$.

Set
$$\hat{y}_1 = y_1^* + \varepsilon$$
$$\hat{y}_2 = y_2^* - \varepsilon$$
$$\hat{y}_3 = y_3^*$$
$$\hat{x}_2 = x_2^*$$

Now $\hat{y}_1 + \hat{y}_2 + \hat{y}_3 = y_1^* + \varepsilon + y_2^* - \varepsilon + y_3^* = y_1^* + y_2^* + y_3^*$

So $(\hat{\boldsymbol{y}}, \hat{x}_2)$ must be feasible.

Max. a Piecewise Linear Concave Function

$$\text{The objective value of } (\hat{\boldsymbol{y}}, \hat{x}_2) \text{ is :}$$

$$\frac{3}{2}\left(y_1^* + \varepsilon\right) + \frac{2}{3}\left(y_2^* - \varepsilon\right) + \frac{1}{4}y_3^* + 2x_2^* = \frac{3}{2}y_1^* + \frac{2}{3}y_2^* + \frac{1}{4}y_3^* + 2x_2^* + \frac{5}{6}\varepsilon$$

$$> \text{objective value of } \left(\boldsymbol{y}^*, x_2^*\right)$$

Therefore, ($\boldsymbol{y}^*$, x^*_2) is not optimal!
This is a contradiction! So it must be that if $y^*_1 < 2$ then $y^*_2 = 0$.

The other properties needed can be shown to hold by similar arguments.

Exercise: Show that if $y^*_1 < 2$ then $y^*_3 = 0$, and that if $y^*_2 < 3$ then $y^*_3 = 0$.

THUS THE NEW LP MODEL IS VALID!

Max. a Piecewise Linear Concave Function

The LP model has optimal solution

$$y^*_1 = 2, \quad y^*_2 = 3, \quad y^*_3 = 0, \quad x^*_2 = 3^1/_3$$

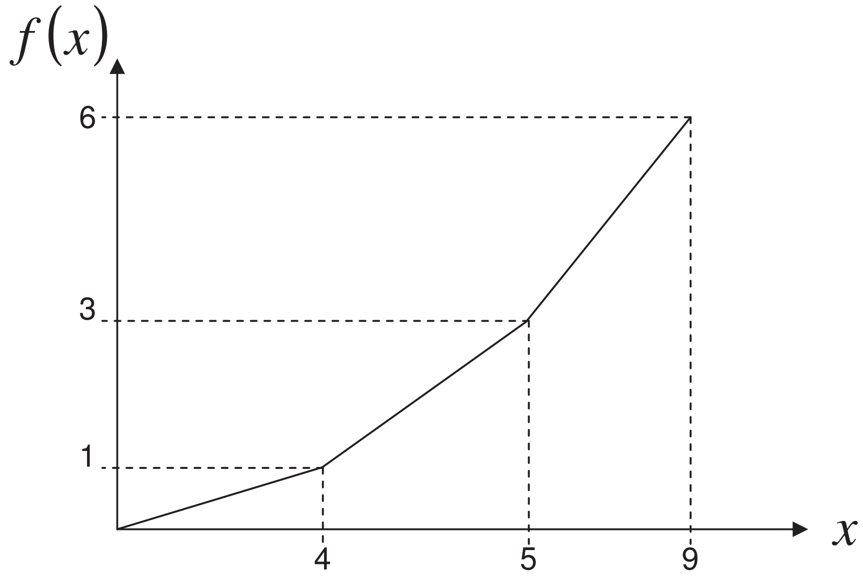
i.e.

$$x^*_1 = 5, \quad x^*_2 = 3^1/_3$$

is the optimal solution to the original problem.

Max. a Piecewise Linear Concave Function

What if *f* was conVEX instead?



If we do the same kind of LP model with $x_1 = y_1 + y_2 + y_3$ in the constraints where

$$0 \le y_1 \le 4,\ 0 \le y_2 \le 3 \text{ and } 0 \le y_3 \le 4$$

we get the optimal LP solution to be

$$y^*_1 = 0, \qquad y^*_2 = 3, \qquad y^*_3 = 2, \qquad x^*_2 = 3^1/_3$$

with LP value $11^2/_3$.

Max. a Piecewise Linear Concave Function

But then $x^*_1 = 5$, $x^*_2 = 3^1/_3$ which has objective value $8^1/_3$ in the original problem!

The LP model is <u>not</u> using the correct objective!

Exercise: Check this by solving the LP.

Hard exercise: What is the optimal solution of the original problem?

Max. a Piecewise Linear Concave Function

In general, a problem maximising a convex function CANNOT be re-modelled as an LP.

Similarly minimising convex function which are piecewise linear can be solved using an LP model, but minimising concave function CANNOT be.

For general functions:

	Minimising	**Maximising**
CONVEX	EASY	HARD
CONCAVE	HARD	EASY
NEITHER	HARD	HARD